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RESEARCH ARTICLE

On the positive fundamental value of money with short-sale constraints

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Abstract This paper is concerned with the pricing of money in a framework with restrictions on trading, under an extension of the standard-asset pricing theory that recognizes both tangible and intangible returns. It is argued that the underlying motivations for demanding money give content to its fundamental value and the bubble component. This approach is illustrated by analyzing the case where no short-sales are allowed, as two examples from the literature are made used to assert that money is a pure pricing bubble. Owing to this setup exhibits technically incomplete financial markets, the fundamental value of money is not uniquely defined over the set of generalized state-price processes. Then, these examples are shown to comprise an extreme case, as money is a pure store of value for the state-prices chosen (i.e., it is a pricing bubble). Instead, the fundamental value of money can be positive for other state-prices, representing the role of money in the trading process. Therefore, money should not be considered the equivalent of a pure pricing bubble.

Keywords Money · Short-sale constraints · Fundamental value · Asset pricing bubbles · (Technically) incomplete financial markets · Financial recognizability and insurance services

JEL Classification Numbers D52 · E41 · G12

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1 Introduction

This paper is concerned with the pricing of money in a framework with restrictions on trading, where *money* is defined as "a security of infinite maturity in positive net supply that pays no physical dividends". Under an extension of the standard asset pricing theory that recognizes both tangible and intangible returns, the underlying motivations for demanding money give content to the fundamental value of money and to its residual bubble component. The latter represents the store of value motive, while the former consists of the intangible services to which the monetary security represents a claim (for example, the services provided by the role of money in the trading process). Section 2 puts forward these arguments.

We illustrate this approach by analyzing two well-known examples in the literature, provided by Kocherlakota (1992) (hereafter K92) and Santos and Woodford (1997) (hereafter SW97), who state such an identification between money and pricing bubbles in a restricted environment where financial traders are not allowed to take short positions. Both examples are discussed in Sect. 3 in the light of the approach followed in this paper. Owing to this constrained environment exhibits technically incomplete financial markets, the fundamental value of money is not uniquely defined over the set of generalized state-price processes. These examples, then, are shown to represent an extreme case in which money is only demanded because of it is a pure store of value; that is, these authors chosen a particular state price process in which money is a pure pricing bubble. We argue instead that for other state-price processes the fundamental value of money is positive, representing the services provided by the role of money in the trading process, denoted by the financial recognizability and insurance services. In the same vein, we reconsider the results in *K92*'s Sect. 2. Finally, Sect. 4 concludes.

2 The positive fundamental value of money with financial constraints

Fiat money was unfold by Samuelson (1958) as a social compact that improved the efficiency of the economy in an intertemporal microeconomics optimization model. The Lerner–Samuelson debate[1] lightened that money in his overlapping generations setup is a pure store of value and yields no other services, consequently fitting into Kaldor (1939)'s definition of speculation.[2] Tirole (1985) placed this discussion into the realm of financial economics: in an overlapping generations model with capital accumulation, money is valued only if its rate of return is equal to the rate of return on capital; therefore, "money has a positive value in spite of the fact that it is intrinsically useless (i.e., its market fundamental

[1] See (Lerner 1959a,b) and Samuelson (1959). See also Rymes (1993).

[2] "Speculation [...] may be defined as the purchase (sale) of a particular good or "asset" with a view to re-sale (re-purchase) at a later date" Kaldor (1939, p.1 and 3). That is, money is demanded in order to exchange it for goods or assets in the future, but plays no distinctive role in such an exchange process.

is zero). In other words there can exist a *bubble* on money" (p. 1499).[3] However, it is not clear whether such an identification between "money" and "rational bubble" will stand up within other non-speculative monetary environments.

In fact, financial economics seems to be the area best-suited for tackling the long-standing challenge of integrating the theory of money and the theory of value (e.g., see Hicks 1935, p. 1–2). However, the Arrow–Debreu rational general equilibrium theory and the standard theory of finance are restricted to take in frictionless environments so that money plays no role and then has no return because of its intrinsic uselessness. Instead, if money enhances individual welfare, both directly or indirectly because it plays a role in the process of trading and provides services of value to participants in the market, asset-return "anomalies" appear regarding the observed preference for holding barren money rather than interest-bearing securities or capital goods (see Townsend 1987). Two implications emerge from this positive monetary demand: i) the asset-pricing of the other securities is affected by the existence of these restrictions on trading (see Lucas 1984);[4] and, ii) money can be considered another security that represents a claim of a stream of dividends;[5] and, thus, it can be valued in the light of the asset-pricing theory.[6] The key issue is how the monetary returns are to be measured. Svensson maintains the view, shared by Tirole (1985, Sect. 6), that the "services of money are endogenously determined by the value of relaxing the [trading] constraint the shadow price of the [trading] constraint." (p. 201).[7]

Therefore, the extension of the frictionless asset pricing theory (e.g., Duffie 1996, or LeRoy and Werner 2001) to constrain financial environments, such as those that cause money to be demanded, has to incorporate these intangible returns. This has relevant consequences. First, if the asset pricing formulae include intangible returns, then the number of state prices at each period will be increased because of the binding trading constraints, one additional shadow price per restriction. As a result, some standard (technical) definitions of *complete* financial markets could not be applied straightforwardly, owing

[3] Subsequent contributions have come from Weil (1987, 1990), O'Connell and Zeldes (1988), and Bertocchi and Wang (1995).

[4] As Townsend (1987) pointed out: "market imperfections or trading difficulties which give rise to money may well have implications for asset prices, [to the extent that] standard asset pricing formulas are in jeopardy." (p. 222 and 236).

[5] According to Svensson (1985), "money is priced in complete symmetry with other assets, once its direct return have been appropriately defined[; and,] once these [monetary] services has been specified, the price of money can be determined by an asset-pricing equation as the price of other assets; only the [monetary] services replace the dividends in the equation." (p. 926 and 920).

[6] Early developments in monetary asset-pricing in general equilibrium are found in Patinkin (1965, Sec. V, ft.18), and LeRoy (1984) for the case of monetary services given by the direct marginal utility of real balances, and Svensson (1985) and Townsend (1987) for the case of monetary liquidity services.

[7] Friedman (1956) and Patinkin (1965, Sect. V) had previously identified the productive services of money with its non-observable non-pecuniary return, but the nature of the services yielded by money and the conditions governing the marginal productivity of money were not very extensively explored in their work.

to complete financial markets at some period will then require a number of non-redundant securities equals to the number of subsequent nodes *plus* the number of the active trading constraints.[8]

Second, the definition of the fundamental value of a security, i.e., the expected discounted value of the stream of its returns, must include not only its tangible dividends, but also its intangible services.[9] This present value and then the asset-pricing bubble, i.e., the difference between the price of the security and its fundamental value, are referred to only a particular state-price process; thus, neither value need be uniquely determined except in the case of complete financial markets, for several state-price processes may exist when financial markets are incomplete. This would be the case, for example, whenever the number of non-redundant securities equals the number of subsequent nodes while some constraint on trading is active.

A third consequence relates to the possibility of a positive fundamental value of money in restricted monetary setups. A main conclusion that can be drawn from SW97 (Theorems 3.1 and 3.3) is that, in frictionless dynamic equilibrium models with symmetric information, asset pricing bubbles on securities in positive net supply occur under rather limited circumstances;[10] consequently, the price of money must be zero (Corollary 3.2). However, when the usefulness of money is socially recognized to circumvent restrictions on trading, so that its demand may differ from a bare store of value, the monetary security constitutes a claim on an infinite stream of (present or future) services represented by the role of money in the trading process. These intangible returns make up the monetary asset-pricing equation and, it is important to realize, they comprise the real value of money, since it is precisely money that gives rise to them when motives for demanding money, other than speculative ones, exist. As a result, the fundamental value of money can be positive, whereas its bubble component can be zero.[11] Yet, few exceptions have examined the conditions under which asset-pricing bubbles may or may not exist in restricted environments.[12] Finally, it should be noted that a distinctive feature of money-pricing is that the value of the intangible monetary returns depends likewise on the path of all sequences of (present and future) prices of money, which has been stressed as a major source of multiplicity in monetary equilibria (Santos 2006) as well as for market fundamentals of money (Tirole 1985, p. 1516). This paper will show a

[8] See Giménez (2003). See also Santos and Woodford (1992, 1996), and Santos (2006).

[9] A usual definition for the fundamental value of a security, i.e., the present value of the "buy-and-hold" portfolio strategy involving such an asset, may not be appropriated for securities that play a role in the trading process and provide non-speculative services.

[10] For example, whenever the discounted value of the resources is infinite, as in the overlapping generations monetary models of Samuelson (1958), Wallace (1980), or Tirole (1985).

[11] The former has been looked at for different monetary services, such as liquidity services (Svensson 1985, Santos and Woodford 1992, Sect. 4, 1996; Santos 2006), trading time savings (Tirole 1985, Sect. 6), reserve requirements services (Giménez-Fernández 1996, Giménez 1998), or short-sale services (Santos and Woodford 1992, Sect. 5).

[12] See Kocherlakota (1992), Magill and Quinzii (1996), Santos and Woodford (1992, 1996), Giménez-Fernández (1996), Giménez (1998), Montrucchio (2004) and Santos (2006).

related source of multiplicity for the fundamental value of money comes from the (technical) incompleteness of financial markets since several schemes of discounting are possible, representing the variety of motives for demanding money, namely the *bubbly view* or the *fundamentalist view* for giving value to money.[13]

3 The positive fundamental value of money: the case of short-sale constraints

To illustrate the controversy surrounding the fundamental value of money, we will focus on the case of financial traders not being allowed to take short positions, for two examples from the literature consider money a pure pricing bubble: K92's example 1, and the non-borrowing constraint example 4.2 provided by SW97, which is rethought as a short-sale restriction economy.[14] For the same deterministic infinite-horizon Arrow–Radner economy where money is the only security whose short sales are forbidden, these authors take for granted that the positive value of money is a pure pricing bubble, whether explicitly: "the asset dividend equals *zero* in every period; hence the asset's fundamental value of money is zero in all periods." (Kocherlakota 1992, p. 251);[15] or by disguising the existence of a financial constraint: "a pricing bubble is possible if borrowing limits are not [the lower discounting value of the aggregate endowment;] there exists valued fiat money, and this is possible because for each household the [zero] borrowing limits effectively bind infinitely often." (Santos and Woodford 1997, p. 40–41). We will show that both assertions might not be true in general.

First, money affords intangible services as long as the no-short-sale constraint is binding. The nature of these intangible returns can be traced by extending to intertemporal trading Brunner and Meltzer's (1971) intuitions regarding the services rendered by money under informational and institutional constrained settings. In environments where high information costs exist on the repayment of debts in securities issued by financial traders, or where no property rights on two-side agreements can be legally defended, short sales may not be accepted by sellers. Hence, a security of social acceptance in positive net supply, such as money, reduces the informational requirements for trading; that is, money is a device for overcoming the adverse selection problem, as it provides financial recognizability and insurance services. The *financial recognizability property* of money stands for the sellers' desire to be paid with money, an object socially

[13] These are, respectively, whether "money is a pure store of value, and does not serve any transaction purpose at least in the long run[, and] is held entirely for speculation;" or, whether "money is held to finance transactions [and, despite m]oney must, of course, be a store of value[,] it is not held for speculative purposes as there is no bubble on money." (Tirole 1985, p. 1517)

[14] SW97 only consider borrowing limits on the agent's negative holdings to avoid Ponzi games. However, in example 4.2 they set the borrowing limits exogenously to zero, which is equivalent to a no-short-sale constraint.

[15] Other authors seem to share this identification. For example, (LeRoy 2005, p.1) begins, "bubbles, such as money, cannot be valued in efficient equilibria in overlapping generations models." Others, at least, do not find it at odds, such as Cassese (2005, p. 530) or (Allen et al. 1993, p. 210).

recognized for exchange purposes, instead of with IOU securities issued by their trading partners, whose debts may not be honored with complete certainty; meanwhile, the *financial insurance service* means that buyers consume more relative to what they would consume in the barter economy because money disconnects what buyers can buy from how they are perceived; in this sense, money acts as consumption insurance.[16] Which of these monetary effects on agents' behaviour is predominant and why money constitutes a social compact are issues beyond the aims of this paper.

Second, financial markets are (technically) incomplete at all dates because there is only one asset in their economy, while a binding no-short-sale restriction exists at each period; consequently, the set of state price processes is not unique in this set-up. Third, the fundamental value of money may be positive, as money represents a claim on an infinite stream of future no-short-sale services.

Finally, since this constrained economy exhibits (technically) incomplete financial markets, a source of multiplicity arises for the fundamental value and the residual bubble component representing the different traders' reasons for demanding money, namely the "financial recognizability and insurance" services and the pure store of value services, respectively. This issue will be analyzed in detail in the following section.

3.1 A reinterpretation of Kocherlakota (1992, Example 1) and Santos and Woodford (1997, Example 4.2)

Next, we will study the pricing of money in an economy where money is the only asset and it is restricted to no-short-selling.

3.1.1 The model

The model follows the main lines depicted by the two-agents two-period-cyclical-endowment economy shown in K92, Ex. 1 and SW97, Ex. 4.2. The framework is a deterministic infinite-horizon short-sale constraint Arrow–Radner economy with two infinitely-lived representative agents, denoted by A and B. The notation is taken from SW97. Agents' preferences over consumption sequences are represented by the same intertemporal additively separable utility function, $\sum_{t=0}^{\infty} \beta^t U(c^h(s^t))$, where the discount factor $\beta \in (0,1)$ is constant and common to both agents, and $U(c)$ is a differentiable strictly quasiconcave, and strictly increasing continuously function on consumption. The temporal structure consists of a sequence of nodes denoted by $\{s^\tau\}$ with a unique initial node s^0 (i.e., $s^0, s^1, \cdots, s^{t-1}, s^t, s^{t+1}, \ldots$). Agent A receives an endowment of

[16] We have adapted the terminology of Berentsen and Rocheteau (2006) to extend Brunner and Meltzer's intuitions to intertemporal trading. While these articles, as well as King and Plosser (1986), focus on whether the use of money decreases informational costs when the quality of goods is uncertain in a barter economy, here the use of money improves agents' welfare intertemporally over an intertemporal credit barter economy where intertemporal agreements have to be guaranteed. This intuition has already been expounded in Ostroy and Starr (1974).

$\omega^A(s^t) = \overline{\omega}$ units of a perishable good when s^t is an even node, and $\omega^A(s^{t+1}) = \underline{\omega}$ units when node s^{t+1} is odd, with $0 \leq \underline{\omega} < \overline{\omega}$; and vice versa for agent B (i.e., $\omega^B(s^t) = \underline{\omega}$ when s^t is an even node, and $\omega^B(s^{t+1}) = \overline{\omega}$ when node s^{t+1} is odd).[17] The aggregate endowment is constant for all s^t: $\omega(s^t) = \underline{\omega} + \overline{\omega}$. There exists only one infinitely-lived security m, in positive net supply, that represents at any node a claim on one unit of the same security at the next node. Free disposal of this security is assumed. Agent B receives, at initial node s^0, the whole initial endowment of monetary security $m^B(s^{-1}) = \widehat{m}^B(s^0) = M$, while agent A receives none (i.e., $m^A(s^{-1}) = \widehat{m}^A(s^0) = 0$).[18]

This is a general equilibrium monetary model. After several operations are done, the model can be expressed in real terms by normalizing the price of consumption good to one. The price of money (i.e., its purchasing power capacity) at node s^t, in terms of the good, is given by $\pi(s^t)$. For the sake of simplicity only agent A's problem is presented, as agent B's problem is the same with $\widehat{m}^B(s^0) = M$ but at every other node: [19]

$$(P_A)\begin{cases} \max_{\{c^A(s^\tau), m^A(s^\tau)\}_{\tau=0}^\infty} \sum_{\tau=0}^{\infty} \beta^\tau U(c^A(s^\tau)) \\ \text{s.t.} \quad \begin{cases} c^A(s^t) + \pi(s^t)m^A(s^t) \leq \overline{\omega} + \pi(s^t)m^A(s^{t-1}) & : \lambda^A(s^t) \\ \pi(s^t)m^A(s^t) \geq -B^A(s^t) & : \mu^A(s^t) \end{cases} \\ \qquad \begin{cases} c^A(s^{t+1}) + \pi(s^{t+1})m^A(s^{t+1}) \leq \underline{\omega} + \pi(s^{t+1})m^A(s^t) & : \lambda^A(s^{t+1}) \\ \pi(s^{t+1})m^A(s^{t+1}) \geq -B^A(s^{t+1}) & : \mu^A(s^{t+1}) \end{cases} \\ \text{given } \pi(s^\tau) \qquad B^A(s^\tau) = 0 \qquad \text{at every } s^\tau \qquad \widehat{m}^A(s^0) = 0 \end{cases}$$

where $\lambda^h(\cdot)$ and $\mu^h(\cdot)$ are agent h's state multipliers of budget and borrowing restrictions. Given that $B^h(s^\tau) = 0$ precludes borrowing, and provided the price of money is positive since agents wish to smooth consumption across periods, this borrowing limit can be thought of as a financial no-short-selling constraint, i.e., $m^h(s^\tau) \geq 0$. First order conditions at s^τ will be

$$\begin{aligned} U'(c^h(s^\tau)) - \lambda^h(s^\tau) &= 0 \\ -\lambda^h(s^\tau)\pi(s^\tau) + \beta\lambda^h(s^{\tau+1})\pi(s^{\tau+1}) + \mu^h(s^\tau)\pi(s^\tau) &= 0. \end{aligned} \tag{1}$$

Because the agents' good endowments are alternate, the strictly quasiconcavity of preferences impels them to smooth consumption along time. At even nodes s^t when agent $h = A$ is "rich," she saves by taking long positions on the existing security; at odd nodes s^{t+1} when poor, she consumes all her savings and the restriction on short sales keeps her from smoothing consumption further, even if it is desired. Consequently, when $h = A$ is "rich", her borrowing limit

[17] Kocherlakota's formulation instead assumes two-cyclical growing endowments (i.e., $\underline{\omega}(1+g)^t$ and $\overline{\omega}(1+g)^t$ where $g > 0$ is a positive growth rate).

[18] In fact, for the stationary case, this initial distribution only affects the equilibrium allocations at initial node s^0.

[19] Unless indicated, we will distinguish even and odd nodes by denoting the set of even nodes as $\{\ldots, s^{t-2}, s^t, s^{t+2}, \ldots\}$, and the set of odd nodes as $\{\ldots, s^{t-1}, s^{t+1}, \ldots\}$.

restriction is not binding (i.e., $\mu^A(s^t) = 0$), while it does bind when "poor" (i.e., $\mu^A(s^{t+1}) \geq 0$); and, vice versa for agent B at every other node.

3.1.2 An Arrow–Radner competitive equilibrium

Definition 1 *An Arrow–Radner competitive equilibrium E is a sequence of consumption allocations, asset allocations, and asset prices* $\{\{c^h(s^\tau), m^h(s^\tau)\}_{h=A,B}, \pi(s^\tau)\}_{\tau=0}^\infty$ *such that:*

(1) $\{\{c^h(s^\tau), m^h(s^\tau)\}_{h=A,B}\}_{\tau=0}^\infty$ *is a solution to agents A's and B's maximization problem, given the equilibrium price sequence* $\{\pi(s^\tau)\}_{\tau=0}^\infty$*; and,*
(2) *goods and financial markets clear:*

$$\sum_{h=A,B} c^h(s^\tau) = \omega(s^\tau), \quad \text{and} \quad \sum_{h=A,B} m^h(s^\tau) = \sum_{h=A,B} m^h(s^{\tau-1}) = M \quad \text{for all } s^\tau.$$

From both agents $h = A, B$ first order conditions for "good" periods (at s^t and s^{t+1}, respectively), and from clearing market conditions, it follows that equilibrium allocations are attained by

$$\left(c^h(s^\tau), m^h(s^\tau)\right) = \begin{cases} (\overline{\omega} - \pi(s^\tau)M, M) & \text{at } s^\tau \text{ "good period" for agent } h\text{; and,} \\ \left(\underline{\omega} + \pi(s^\tau)M, 0\right) & \text{at } s^\tau \text{ "bad period" for agent } h \end{cases};$$

whereas the law of motion of security price is determined by

$$\pi(s^\tau)U'\left(\overline{\omega} - \pi(s^\tau)M\right) = \beta\pi(s^{\tau+1})U'\left(\underline{\omega} + \pi(s^{\tau+1})M\right). \tag{2}$$

3.1.3 No-arbitrage conditions and financial complete markets

Santos and Woodford (1992, Sect. 5), state a definition of no-arbitrage opportunities in this framework. For this example the following version is presented. It is said that *no-arbitrage opportunities exist* at node s^t for the price process $\{\pi(s^\tau)\}_{\tau=0}^\infty$, if for all $m \in \Re$ the conditions

$$\begin{aligned} \pi(s^{t+1})m &\geq 0 \quad \text{for } s^{t+1} \text{ successor of } s^t \\ \pi(s^t)m &\leq 0 \\ m &\geq 0 \end{aligned}$$

imply

$$\begin{aligned} \pi(s^{t+1})m &= 0 \quad \text{for } s^{t+1} \text{ successor of } s^t \\ \pi(s^t)m &= 0. \end{aligned}$$

They also show the resulting existence of a state-price process, as a direct consequence of the Farkas–Minkowski lemma. For this deterministic example, we need only to demonstrate that the non-existence of arbitrage opportunities at any node s^t implies the existence of a set of *state-price vectors* $\{a(s^T)\}$ with $a(s^T) > 0$ for any s^T successor of s^t, and additional *shadow prices* $\{r(s^T)\}$, which emerge from the existence of the no-short-sale constraint, with $r(s^T) \geq 0$ for all s^T successors of s^t, which verify (as a consequence of the Farkas–Minkowski lemma): $r(s^T)\pi(s^T)m^h(s^T) \geq 0$, for every $h = A, B$, and with equality in the case of $m^h(s^T) = 0$; and, such that

$$a(s^t)\pi(s^t) = r(s^t)\pi(s^t) + a(s^{t+1})\pi(s^{t+1}). \tag{3}$$

It is worth noting that this is the asset-pricing equation for the value of a claim of monetary no-short-sale services, and it is analogous to the first order condition for monetary security (1). The nature of these intangible returns, which were termed as financial recognizability and insurance services, has been described in the previous section.

Finally, we characterize the financial markets for this constrained setup. At each period, the state price system above defined is *unique* (up to an arbitrary normalization $a(s^0)$) if, and only if, for every node s^t the number of independent securities with no limits on short-sales is equal to the number of immediate successors of s^t. We can then say that *financial markets* are *complete* at node s^t; otherwise, financial markets are *(technically) incomplete* (see Giménez 2003). In the example studied here markets are (technically) incomplete because agents cannot smooth consumption further by transferring all the wealth they want from "bad" nodes to future "good" ones.[20]

3.1.4 The fundamental value of money

The *fundamental value of the monetary security* at any period s^t is the (expected) discounted sum of the stream of the future financial-recognizability-and-insurance services yielded when the no-short-sale constraint is binding:

$$f_\pi(s^t) \equiv \frac{1}{a(s^t)} \sum_{\tau=t}^{\infty} r(s^\tau)\pi(s^\tau),$$

where $\{a(s^t), r(s^\tau)\}_{\tau=0}^{\infty}$ is the generalized state-price process, with $a(s^t) > 0$ and $r(s^\tau) \geq 0$ for all s^τ. In fact, this definition could be thought of as the infinite forward iteration of asset pricing Eq. (3). If the price of money is higher than its fundamental value at node s^t, there is a *monetary pricing bubble* at node s^t, i.e., $\sigma_\pi(s^t) = \pi(s^t) - f_\pi(s^t)$.

Next, we present a result that rules out pricing bubbles for the monetary security. This proposition could be thought of as a simplified result similar to that of Santos and Woodford (1992, Theorem 5.1), which was not

[20] To complete the financial markets, it would be required an additional security that is allowed to take short positions (i.e., $z^h(s^\tau) \geq -\infty$ at every s^τ), or that preserve the total aggregate individual wealth from being negative: $q(s^t)z^h(s^t) + \pi(s^t)m^h(s^t) \geq 0$ with $m^h(s^t) \geq 0$, where $q(s^t)$ is the price of the new security.

proven there.[21] It states that for securities of finite maturity or in positive net supply, if the discounted value of aggregate endowment is finite then there will exist a discounting scheme for which the price of the security is equal to its fundamental value.

Proposition 1 *Let E be an Arrow–Radner competitive equilibrium with positive price of money. If the transversality condition*

$$\lim_{T\to\infty} a(s^{t+T})\pi(s^{t+T}) = 0 \tag{4}$$

holds, then $f_\pi(s^t) = \pi(s^t)$. □

Proof This proposition works provided that the increase in the price of money is lower than the increase in the economy's rate of return, i.e., condition (4). The proof comes from iterating (3) forward, so we may find

$$a(s^t)\pi(s^t) = a(s^{t+T})\pi(s^{t+T}) + \sum_{\tau=0}^{T-1} r(s^{t+\tau})\pi(s^{t+\tau}).$$

If (4) holds, then as T goes to infinity,

$$\pi(s^t) = \frac{1}{a(s^t)} \sum_{\tau=t}^{\infty} r(s^\tau)\pi(s^\tau) \equiv f_\pi(s^t). \qquad \square$$

Proposition 1 states that, as long as condition (4) holds, all agents will demand money for its no-short-selling motive. Accordingly, the fundamental value of money will equal its price and, then, no pricing bubble on money will exist, i.e., $\sigma_\pi(s^t) = 0$ at any period s^t. This result differs substantially from the conclusions arrived in the literature. In the same setting, the positive value of money has been identified with a pure pricing bubble, whether explicitly (see K92, p. 251), or disguised under the existence of a financial constraint (see SW97, p. 40–41). Some comments are in order. First, Proposition 1 only states that there does not exist a monetary pricing bubble for those state price processes verifying condition (4). Owing to financial markets being (technically) incomplete, several discounting schemes are possible and not all of them will necessarily verify such a condition. So monetary pricing bubbles may exist in terms of *some* of these state price processes.

Second, Kocherlakota seems to embrace the "bubbly view" of giving value to money, as he considers that the monetary security yields only speculative services. Consequently, the fundamental value of money is explicitly assumed

[21] Related proved results are those given by SW97 (Theorem. 3.1) for borrowing limits; by Santos and Woodford (1996, Theorem 4.1), and Santos (2006, Theorem 4.1), for cash-in-advance constraint; and by Giménez-Fernández (1996, Theorem 4.1), Giménez (1998, Theorem 4.1) for reserve requirements.

to always be zero (i.e., $f_\pi(s^t) \equiv 0$ for all s^t). In contrast with this view, we have argued that money may provide intangible services. Therefore, we may reconsider the results found in K92's Sect. 2 in the light of this approach. His statement "the asset price has no bubble if and only if [(4) holds]" (K92, p. 250) is formalized by our Proposition 1. In addition, his two necessary conditions for determining when bubbles can exist in economies where agents face short-sale constraints are still valid. The first condition states that "bubbles can only exist if individuals cannot permanently reduce their holdings of the assets[, so that] short sales constraint must be binding infinitely often if bubbles exist". (p. 250). This is formalized as follows,

Proposition 2 [K92, Proposition 3] *In a finite interior Arrow–Radner competitive equilibrium E with a positive pricing bubble in the asset price,* $\liminf_{T\to\infty} m^h(s^T) = 0$ *for* $h = A, B$. □

The second condition shows that "[if] the per capita supply of the asset is not equal to 0[,$\cdots$] in any finite interior equilibrium with a positive bubble in the asset, the growth rate of some agent's endowment must be at least as large as the rate of interest." (p.250-1). Indeed, it is easy to see that *SW97*'s example 4.2, rethought of as an economy with short-sale restrictions, is now a direct consequence of the following proposition,

Proposition 3 [*K92*, Proposition 4] *In a finite interior Arrow-Radner competitive equilibrium E with a positive pricing bubble in the asset price, for any agent* $h = A, B$ *such that* $m^h(s^T)$ *has no limit as T goes to infinity, it is verified that*

$$\sum_{\tau=t}^{\infty} a(s^\tau)\omega^h(s^\tau) = +\infty.\text{"} \tag{5}$$

□

These two propositions, however, only "provide necessary conditions for the existence of bubbles in (this constrained economy)." (K92, p. 251), but they are not sufficient conditions. For example, despite SW97's and K92's assertion that an ingredient for the existence of pricing bubbles is that agents cannot permanently reduce their holdings of the monetary asset,[22] the result that follows below demonstrates that this claim is not true: even with that ingredient pricing bubbles may not exist if a transversality condition holds.

Corollary 4 *Let E be an Arrow–Radner competitive equilibrium with positive price of money, and such that* $\lim_{T\to\infty} m^h(s^T)$ *does not exist for any agent* $h = A$, *B. If the transversality condition (4) holds, then* $f_\pi(s^t) = \pi(s^t)$. □

The proof is a direct consequence of Proposition 1.

[22] Respectively, "a pricing bubble is possible [...] because for each household the [zero] borrowing limits effectively bind infinitely often" (SW97, pp. 40–41), and "[one of the] two ingredients [...] essential for bubbles in the economy *E*[, jointly with condition (5), is that] no agent can permanently reduce his holdings because of the short sales constraint that binds every other period". (K92, p. 252)

3.1.5 The multiplicity of the fundamental value of money under (technically) incomplete financial markets

As pointed out, the multiplicity of monetary equilibria[23] and of the market fundamentals of money stem from the feature that the value of its intangible returns depends likewise on the whole path of the monetary prices. Next, we will show that a related source of multiplicity in the fundamental value of money comes from the (technical) incompleteness of financial markets since several schemes of discounting are possible, representing the varying motives for demanding money, namely the *bubbly view* or the *fundamentalist view* for giving value to money (see Tirole 1985, p. 1517).

To illustrate this issue, we will focus on the stationary equilibrium with a constant non-zero price of money; that is, $\pi(s^\tau) = \pi(s^{\tau+1}) = \overline{\pi}$ for all periods s^τ. In this case, the dynamic equation (2) is transformed into

$$U'\left(\overline{\omega} - \overline{\pi}M\right) = \beta U'\left(\underline{\omega} + \overline{\pi}M\right).$$

Because of the monotonicity of preferences, there exists an stationary price of the security, which is bounded $0 \leq \overline{\pi} < (\overline{\omega} - \underline{\omega})/2M$. Given that financial markets are (technically) incomplete, several generalized state-price processes $\{(a(s^\tau), r(s^\tau))\}_{\tau=0}^{\infty}$ are compatible with no-arbitrage conditions.[24] Each generalized state-price process defines different fundamental values and pricing bubbles for money, each representing the different trader's reasons for demanding money, namely its *financial-recognizability-and-insurance services* and the *pure store of value services*, respectively. A direct consequence of Proposition 1 is that whenever condition (4) holds, all agents demand the monetary security exclusively for its (non-speculative) no-short-sale services.

Proposition 1, however, does not preclude the existence of state-price processes for which the limit in condition (4) does not converge to zero, where all agents that demand the monetary security exclusively do it for its store of value motive. This is the case of the generalized state-price process $\{a(s^\tau), r(s^\tau)\}_{\tau=0}^{\infty}$ selected by SW97, Ex.4.2, where $a(s^\tau) = 1$ and $r(s^\tau) = 0$ for all s^τ. This particular process implies a zero fundamental value for the monetary security, $f_\pi(s^\tau) = 0$ for all s^τ as money does not yield intangible returns. And then money becomes a pure store of value (i.e., a pure pricing bubble) as it is plainly demanded for speculative motives, $\sigma_\pi(s^\tau) = \pi(s^\tau)$ for all s^τ. We should note that Proposition 1 does not apply here, as (4) does not hold, nor does Santos and Woodford (1992, Theorem 5.1) apply because at any s^t "aggregate endowment is not bounded

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23 For example, if $U(c) = ln(c)$, under the assumption $\beta\overline{\omega} > \underline{\omega}$, there may exist infinite asymptotically autarky equilibria that differ from the stationary one.

24 For instance, $\left\{\left(a(s^\tau), r(s^\tau)\right)\right\}_{\tau=0}^{\infty} \equiv \left\{\left(\beta^\tau, \beta^\tau(1-\beta)\right)\right\}_{\tau=0}^{\infty}$. Also, we may find two different state-price processes from the agents' multipliers, $(a(s^\tau), r(s^\tau)) = (\beta^\tau\lambda^h(s^\tau), \beta^\tau\mu^h(s^\tau))$ for $h = A, B$. That is, $\left\{\left(a(s^\tau), r(s^\tau)\right)\right\}_{\tau=0}^{\infty} \equiv \left\{(\beta, 0), (\beta, \beta(1-\beta^2)), (\beta^3, 0), (\beta^3, \beta^3(1-\beta^2)), \ldots\right\}$; and, $\left\{\left(a(s^\tau), r(s^\tau)\right)\right\}_{\tau=0}^{\infty}$ $\equiv \left\{(1, (1-\beta^2)), (\beta^2, 0), (\beta^2, \beta^2(1-\beta^2)), (\beta^4, 0), (\beta^4, \beta^4(1-\beta^2)), \ldots\right\}$.

by a portfolio trading plan," i.e., $\sum_{\tau=t}^{\infty} a(s^\tau)\omega(s^\tau) = +\infty$.[25] Instead, given that a pricing bubble exists, Proposition 3 holds because the agent's holding of securities does not converge. Consequently, SW97, Ex. 4.2, is converted into a particular case of the analysis presented here.

4 Conclusions

This paper is concerned with the pricing of money in a framework with restrictions on trading, under an extension of the standard asset-pricing theory that recognizes both tangible *and* intangible returns. We illustrate this approach by examining two examples with short-sale constraints, provided by K97 and SW97. Unlike these authors, who consider money to be a pure store of value, we have argued that the monetary security may play a role in the trading process by providing *financial recognizability and insurance* services. These intangible returns comprise the fundamental value of money. Hence, we suggest that a usual definition for the fundamental value, i.e., the discounted value of a buy-and-hold portfolio strategy involving the asset in question, may be inappropriate for an asset that is used to facilitate transactions.

One might consider that the issue treated in this paper is a semantics issue: what some authors have called "positive rational pricing bubble," in this paper we have termed "positive fundamental value," but no consequences were observed for the competitive equilibrium. However, the Walrasian rational general equilibrium analysis requires a clear specification of the underlying motives for agents' to demanding for and supplying of commodities and securities, as must also be true for the monetary security.[26] This is the advantage of the extension of the standard asset-pricing theory followed in this paper, as it leads to further insights into the basis of monetary theories. In a way, this approach could be thought of as a pricing theory of money. Understanding the reasons for demanding money in each particular monetary environment will be of great help in assessing the robustness of monetary models after the introduction of more productive securities. For example, in the setting studied in Sect. 3 where money is the only asset in the economy, the monetary security has a positive price, and a binding short-sale constraint may provide a positive fundamental value. However, this is a very weak theory of money, since agents would be more interested in a more productive security once it is introduced into this economy, so that the intangible monetary services vanish. Even in this case, money could have a positive price but then we could assert that it is demanded exclusively

[25] This is a direct consequence of the fact that condition (4) does not hold, as well as any agent h at any node s^t has a portion of aggregate endowment; that is, $\omega^h(s^\tau) \geq \varepsilon\omega(s^\tau) > \varepsilon\overline{\omega} > \varepsilon\pi(s^\tau)M$, here last inequality comes from the positive consumption of agents' "good period" as their holdings are always positive.

[26] This distinctive feature sharply differs from the ad-hoc aggregate monetary macroeconomic models that include a monetary demand function that specifies several reasons for demanding money –such as precautionary, transactional, and speculative motives–, despite its loose microeconomic basis.

due to speculative purposes, as only store of value would be of interest for some participants in the financial markets. Other more robust monetary theories are ready to be examined using this same methodology.

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